Module 14: THE INTEGRAL

Exploring Calculus

Part I Approximations and the Definite Integral

It was known in the 1600s before the calculus was developed that the area of an irregularly shaped region could be approximated by dividing the region into small pieces and computing the total area of the pieces. By choosing more and more pieces, the error between the total area of the pieces and the area of the region became less and less. We credit the ancient Greeks with the discovery of this method.

Before 1850 mathematicians, beginning with Newton and Leibniz, understood the connection between the integral and the area under a curve. The definition of the definite integral used today, which expresses the area of the region between a certain class of functions and the x-axis, as the limit of a specially selected approximating sum, is attributed to the great mathematician Friedrick Riemann, c. 1850. In this module, we will discover more about the relationship between the definite integral and the area under a curve.

We approximate the area under curves by breaking that area up into a number of pieces with bases on the x-axis and sides parallel to the y-axis. Depending on the tops chosen, the pieces become either rectangles, trapezoids, or regions whose tops are made of portions of parabolas--referred to respectively as Upper and Lower Rectangles; the Trapezoid Rule; and Simpson's Rule.

Critical Thinking Questions

1. What are the upper and lower sums for the region above the x-axis

a. On the interval [-2,5] ?  Choose the function y = x + 2 , on [-2, 5] , with 20 rectangles (choose 300 points to plot). (Ula_.bat)

b. On the interval [0,5] ?  Choose the function y = x + 2 , on [0, 5] , with 20 rectangles (choose 300 points to plot). (Ulb_.bat)

2. Use area formulas with which you are familiar to find the area under y = x + 2 , above the x-axis, on the interval

a. [-2, 5]

b. [0, 5]

3. Which sum in Q.1a--b was correct, the upper or lower?

4. a. How could you use *both* the upper and lower sums to get a better approximation to the area under a given function?

b. What happens to the approximation using the upper or lower sum if you increase the number of rectangles?

5. Estimate the area under the curve using the method you found in Q.4a. Choose the function $y = x^5 + 100$, on [0, 5], with 20 rectangles (choose 300 points to plot).

You will now approximate the area under a curve when the regions are *trapezoids.*

6. Display the graph for the function $y = x^5 + 100$, on [0, 5], with 5 intervals. **(Trapa_.bat)**

a. What is the trapezoidal approximation to the area under the curve?

b. If you increase the number of intervals selected, what do you think will happen to the error?

c. What are the approximations to the area with 10 and with 20 intervals? How is the answer with 20 intervals related to the answer to Q.5? Explain why this is so.

Display the graph for the function $y = x^5 + 100$, on [0, 5], with 10 intervals. **(Trapb_.bat)**

Display the graph for the function $y = x^5 + 100$, on [0, 5], with 20 intervals. **(Trapc_.bat)**

d. How many intervals does it take to make the error less than 5?

Display the graph and choose the function and interval necessary to answer the question above.**(Trapd.bat)**

e. If you change the function to $y = x^6$, $y = x^7$ and then $y = x^8$ on the same region, why do you think the **percent error** becomes *increasingly* larger than it was for $y = x^5 + 100$,on [0, 5]. (Consider whether this is due to the regions having larger *area* or whether it is due to the *degree* of the function changing.)

7. Explain why when five intervals are selected with the trapezoid rule, the error is larger than when a larger number of intervals are selected.

8. Suppose the tops of the regions approximating the area are parabolic.

a. How do you expect the error to compare with the error using the previous methods?

b. Why do you think this is so?

You will now work with Simpson's rule, which approximates the area under a curve with regions whose tops are *parabolic*.

The curve is shown on the graph in red, and the approximating regions with parabolic tops are shown in yellow.

Choose $y = x^5 + 100$, $0 \le x \le 5$ with the number of intervals equal to 6. In this case [0, 5] is in six pieces divided by the seven x-values $x = 0, 5/6, 5/3, 5/6, 10/3, 25/6,$ and 5

The number of parabolas drawn is the number of intervals divided by 2. The first parabola is drawn through the y-values corresponding to the first three x-values, and the second parabola is drawn through the y-values corresponding to the third through the fifth x-value, and the third parabola is drawn through the y-values corresponding to the fifth through the seventh x-values. So, if n equals the number of intervals, there will be parabolic tops and they will touch each other at $n/2 - 1$ places. Thus, on the interval [0,5], when n = 6, there are _____ parabolas drawn, and they touch at the x-values _____ _____.

Your answers to questions 9 & 10 below, should be accurate to the fourth place to the right of the decimal point.

9. a. The area when n = 6 is _____ , and the error is _____ . **(Simpa.bat, n = 6)**

b. The area when n = 10 is _____ , and the error is _____. **(Simpa.bat, n = 10)**

c. The area when n = 20 is_____ , and the error is _____. **(Simpa.bat, n = 20)**

d. The area when n = 60 is _____ , and the error is _____. **(Simpa.bat, n = 60)**

10.  Choose the function $y = x^7 + 5000$, $0 \le x \le 5$.

a. The area when n = 6 is _____, and the error is _____. **(Simpb.bat, n = 6)**

b. The area when n = 10 is _____ , and the error is _____. **(Simpb.bat, n = 10)**

c. The area when n = 20 is _____ , and the error is _____. **(Simpb.bat, n = 20)**

d. The area when n = 60 is _____ , and the error is _____. **(Simpb.bat, n = 60)**

11. Is the fact that the percent error for Q.10 is larger than the percent error for Q.9 because the regions have larger area, or is it because of some other reason? Explain.

12. Use your text to find formulas to predict the maximum error using the Trapezoid Rule or Simpson's Rule. For each situation in Qs. 6, 9 and 10, determine the maximum predicted error.

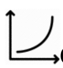
Show your work.

Max. predicted error Q.6	a.	c.	c.	
Max. predicted error Q.9	a.	b.	c.	d.
Max. predicted error Q.10	a.	b.	c.	d.

Are your computer estimates less than the maximum predicted error?

13. Rank the approximation methods Upper and Lower Rectangles, Trapezoid Rule and Simpson's Rule in the order of their accuracy.

You will now work with the Riemann Sum which, while it is *not useful as an approximation method*, will be used to define the **definite integral** later in this module.

Choose the function $y = x^5 + 100, 0 \le x \le 5$ with 300 intervals and with 20 , 50 , and then 100 rectangles. **(Riem_.bat)** (**Repeat each selection at least several times.** *Note that this method is not useful for its accuracy, but is used to set up a proof interrelating the sums of areas of regions and the definite integral*.**)**

14. a. Do the widths of the rectangles in each case follow a regular pattern?

b. Does each rectangle hit the curve in the same place, e.g., left, middle, right, etc.?

c. How is this method of approximating the area under the curve different than the method of Up*per / Lower rectangles* ?

d. Explain how the area under a curve is estimated using a Riemann Sum.

15. What do you think would happen to the error using the Upper/Lower rectangles, Trapezoid, Simpson, or Riemann approximation, if the number of intervals chosen became larger and larger?

16. As the number of intervals becomes larger and larger (as n approaches inf), the size of each interval becomes _____ . In the case of the Riemann approximation, the size (on average) of the _____ interval becomes _____ when

the number of intervals increases.

17. Let Δx_i = the width of the interval [x $_{i-1}$, x $_i$] , that is, the width of one of the Riemann rectangles.

a. As the x_i change, the Δx_i _____ , or we may say the intervals are _____ wide.

b. The (*) in f (x_i^*) denotes that the x-value is _____ placed between x $_{i-1}$ and x $_i$.

c. f (x_i^*) represents the _____ of the approximating region on the interval [x $_{i-1}$, x $_i$] .

d. f (x_i^*) Δx_i represents the _____ of the region above the interval [x $_{i-1}$, x $_i$] .

e. The symbol $\sum_{i=1}^{n} f(x)_i^* \Delta x_i$ represents the _____ of the products f (x_i^*) Δx_i starting with i = _____ and ending with i = _____.

f. The symbols $\lim_{max\ \Delta x_i \to 0}$ means that the width of the _____ interval goes to_____ .

g. Among the contributions of F. Riemann was his choice of a very general type of sum so that

$$\lim_{max\ \Delta x_i \to 0} \sum_{i=1}^{n} f(x)_i^* \Delta x_i$$

is easily calculated--without using limits--for certain (integrable) functions; and is equal to the _____ _____ under the curve f (x), above the x-axis, between x = a and x = b, when f (x) $\geq$ 0 .

18. We now introduce the new symbol $\int_a^b f(x)dx$. called the **definite integral** of the function f (x) on the interval [a,b], and *define* it so that (for the partition of [a , b] as in Q.17g)

Fill in the blanks below.

$$\int_a^b f(x)dx \;\_\_\_\_\_\; \lim_{max\ \Delta x_i \to 0} \sum_{i=1}^{n} f(x)_i^* \Delta x_i$$

that is, the integral of the function y = f (x) between the x-values a and b _____ the limit of the Riemann Sum as the mesh of the partition (the length of the largest interval) goes to zero.

In Part II of this module, you will use $\int_a^b f(x)dx$ to calculate areas of regions, using a very easily applied theorem.

19. Suppose f (x) is below the x-axis between x = a and x = b .

a. What do you think will happen to the area estimate using one of the approximating rules?

Hint: *Try Simpson's rule on the computer.* ***(Simp_.bat)***

b. Why does this happen?

20. Consider the region between the x-axis and the function $y = -x + 2.5$ on the interval [0 , 5]

a. What do you predict the computer approximation to the area will be?

b. Verify your answer to Q.20a using one of one the methods Upper / Lower Rectangles, Trapezoid Rule, or Simpson's Rule .

c. Explain your answer.

d. What is the area?

The Mean Value Theorem

In this module you will investigate whether a certain property always holds true for continuous functions on a given interval. By analyzing a variety of displayed functions exemplifying this property, you will determine when this result--called the Mean Value Theorem--is valid.

Use and accept the default left and right endpoints, the default values of **a** and **b** and the default function f (x). The graph will draw the given function with a secant line drawn in **blue** through the points (a, f (a)) and (b,f (b)).

Critical Thinking Questions

1. How is the slope of the **red line** that touches the graph of $y = x^3 - 3x^2$ related to the slope of the **blue secant line**?

2. The **red line** is __________ to the graph of the function.

3. What are the coordinates of the two points through which the **blue secant line** passes?

4. What is the slope of the secant line?

5. Since the **red line** is __________ to the graph of f (x), we could determine its slope by finding the ___________ of the function f (x) at the __-value of the point where the **red line** hits the graph.

6. Therefore, the slope of the secant line is __________ to the slope of the __________ to the graph of f (x) at some __-value between **a** and **b**. (This result is called the **Mean Value Theorem**.)

7. a. For the above result to be true, must a = - b?

b. Explain in words why (or why not) this result *always* holds true for continuous functions on some interval [a,b]? (Draw picture(s) to justify your conclusion.)

Skill Exercises

1. For the given values of **a** and **b** and the given function in the CTQ above, set up and solve an equation to determine the x-value where the **red line** must intersect the graph for it to be parallel to the **blue line**. (Hint: See CTQ 6. above.)

2. Repeat the above exercise for a continuous function of your choosing.

3. State the Mean Value Theorem for a function f(x), with secant line through the points (a, f (a)) and (b, f (b)).

Part II An Application of the Integral to Rocket Propulsion

The second stage of a two-stage rocket is ignited at time, t = 0 , and travels vertically upward. When the second-stage rocket is 2000 meters above the ground, its velocity is 250 meters per second *(m/s)*. The rocket at this time is programmed to undergo a *nonconstant* acceleration, a = 12 t m/s^2, for a *burn-time* of 5 seconds *(s)* . The following exercises will direct you toward finding the maximum velocity and subsequent maximum altitude above ground level, **h** , which the rocket obtains.

1. You will first determine the equation for the rocket's acceleration, a (t) .

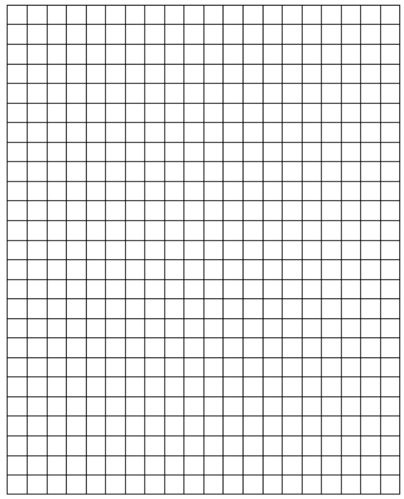
a. What is a (t) for t < 5 ?

b. Is there acceleration provided by the rocket engine after t = 5s ?

c. What causes acceleration after t = 5s ?

d. What is a (t) for t > 5s ?

e. Sketch a graph of a (t) on the grid below.



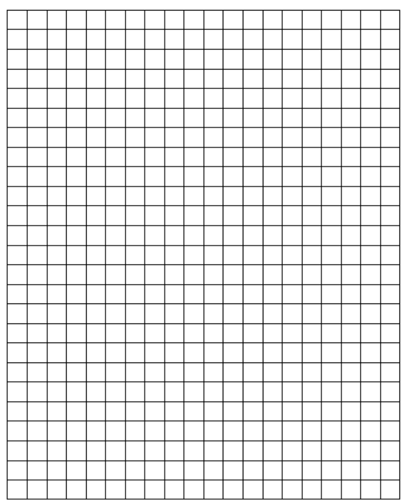
2. You will now determine the equation for the velocity of the rocket, v (t) .

a. v (t) is the of _____ a (t) .

b. v (t) = _____. **Determine this for both t <= 5 and for t > 5 .**

c. The area under a (t) , 0 <= t <= 5 , equals the_____ in v (t) , 0 <= t <= 5. This value is _____ .

d. Graph v (t) on the grid below and use the values from Q.2c as part of the graph's label.



3. You will now determine the equation for the height of the rocket, h (t) .

a. h (t) is the_____of.

b. h (t) =_____.   **Determine this for both t <= 5 and for t > 5 .**

c. The area under v (t), 0 <= t <= 5, equals the_____ in h (t) , 0 <= t <= 5. This value is _____ .

4. You will now determine the maximum altitude.

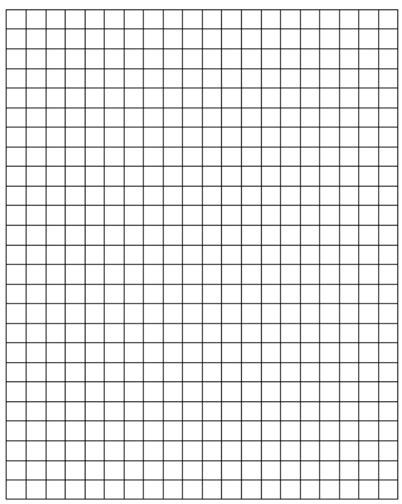
a. What is the maximum height obtained by the rocket?

b. What is the velocity at the maximum altitude?

c. What is the acceleration at the maximum altitude?

d. The area under v (t) for t > 0 equals the _____ in h (t) for t > 0.

e. Graph h(t) on the grid below and use the value from Q.4d as part of the graph's label.



Bonus question for this module.

You are travelling on the interstate in your car at 60 mph when you see a car coming up on you in your rearview mirror. Suppose you know that it is travelling at 70 mph and that your car is capable of a constant acceleration of 5 feet per s^2 . You hate people passing you, so you stomp on your accelerator as soon as you see the car gaining on you.

Assume that you accelerate at your maximum capability the whole time.

1. How far behind you must the other car be when you start accelerating so that it will just come even with you, but not pass you?

2. How long will you be accelerating?

3. How fast will you be traveling when the other car is even with you?

Click here to take an Assessment Test for this module.

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